Defn The m-th Poincaré series of wt k for Γ is the function

$$P_m(z) = P_m^k(z) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (cz+d)^{-k} \underbrace{e^{2\pi i m \gamma z}}_{:= e(m\gamma z)}$$

$$P_m^k(z) = \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)=1 \\ c>0 \text{ or } (c,d)=(0,1)}} (cz+d)^{-k} e^{2\pi i m\left(\frac{az+b}{cz+d}\right)}$$

Since $\left|\frac{e^{2\pi i m \gamma z}}{(cz+d)^k}\right| = \frac{e^{-2\pi m \operatorname{Im}\gamma z}}{|cz+d|^k} \leq \frac{1}{|cz+d|^k}$

for $y>0$, abs. and unif conv. on compacta of Poincare series follows from that of Eisenstein series.

One can show directly that $P_m^k(z)$ are cusp forms i.e. $\lim_{y\to\infty} P_m^k(iy)=0$

<u>Lemma 5.1</u> $P_m^k(z) \in S_k$

The subsum over terms with $c\neq 0$ can be estimated in abs. value

$$\sum_{\substack{(c,d)=1 \\ c\neq 0}} \left|\frac{e^{2\pi i m \frac{az+b}{cz+d}}}{(cz+d)^k}\right| \leq \sum_{\substack{c\neq 0 \\ d\in\mathbb{Z}}} \frac{1}{|cz+d|^k}$$

$$= \sum_{\substack{c\neq 0 \\ d\in\mathbb{Z}}} \frac{1}{\left((cx+d)^2+(cy)^2\right)^{k/2}}$$

$$\sum_{d\in\mathbb{Z}} \frac{1}{\left((cx+d)^2+(cy)^2\right)^{k/2}} \ll \int_{\mathbb{R}} \frac{dt}{\left(t^2+(cy)^2\right)^{k/2}}$$

$$\leq \frac{1}{|cy|^{k-1}} \int_{\mathbb{R}} \frac{dt}{(t^2+1)^{k/2}} \leq \frac{K}{y^{k-1}} \frac{1}{|c|^{k-1}} \qquad k>2$$

Hence the sum over $c \neq 0$ is bounded by a constant multiple of $1/y^{k-1}$ which goes to 0 as $y \to \infty$ for $k>2$

The subsum with $c=0$ is just $e^{2\pi i m z}$ which goes to 0 as $y \to \infty$ (since $e^{2\pi i m b} = 1$)

Thus $P_m(z) \in S_k(\Gamma)$

$\boxtimes$

We'll see this also via its Fourier expansion

We drop k as it is fixed

$$P_m(z) = \sum_{n=0}^{\infty} a_m(n) q^n$$

with $a_m(n) = \int_0^1 P_m(z) e^{-2\pi i n z} dz$

$$= \int_0^1 \sum_{\substack{c=0 \\ d=1}} d^{-k} e^{2\pi i m z} \cdot e^{-2\pi i n z} dz$$

$$+ \int_0^1 \sum_{\substack{c \neq 0 \\ (c,d)=1 \\ d \in \mathbb{Z} \\ c>0}} (cz+d)^{-k} e^{2\pi i m \gamma z} e^{-2\pi i n z} dz$$

The first integral gives δ_{mn} $\quad$ 5

since $$\int_0^1 e^{2\pi i(m-n)z}\,dz = \begin{cases} 1 & \text{if } m=n \\ 0 & \text{otherwise} \end{cases}$$

2nd part is

$$\int_0^1 \sum_{\substack{c\neq 0 \\ d\in\mathbb{Z} \\ c>0 \\ (c,d)=1}} (cz+d)^{-k}\, e(m\gamma z - nz)\,dz$$

Fix $c\neq 0$, write d modulo c as

$$d = lc + d', \quad l\in\mathbb{Z}, \quad d' \bmod c, \quad (c,d')=1$$

Then

$$a_m(n) = \delta_{mn} + \sum_{\substack{c>0 \\ d' \bmod c \\ (c,d)=1}} \sum_{l\in\mathbb{Z}} \int_0^1 (c(z+l)+d')^{-k}\, e\left(m\frac{az+b}{cz+d} - nz\right) dz$$

Now using

$$\gamma z = \boxed{\frac{az+b}{cz+d} = \frac{a}{c} - \frac{1}{c(cz+d)}} = \frac{a}{c} - \frac{1}{c(c(z+l)+d')}$$

Hence

$$a_m(n) = \delta_{mn} + \sum_{\substack{c>0 \\ d \bmod c \\ (c,d)=1}} \sum_{l\in\mathbb{Z}} \int_0^1 (c(z+l)+d)^{-k}\, e(-nz)\, e\left(m\left(\frac{a}{c} - \frac{1}{c(c(z+l)+d)}\right)\right) dz$$

let $z = z + l$ to get

$$q_m(n) = \delta_{mn} + \sum_{\substack{c>0 \\ d \bmod c \\ (c,d)=1}} \sum_{\ell \in \mathbb{Z}} \int_{\ell}^{\ell+1} (cz+d)^{-k} e(-nz)\, e\left(\frac{ma}{c}\right) e\left(\frac{-m}{c(cz+d)}\right)$$

$$= \delta_{mn} + \sum_{\substack{c>0 \\ d \bmod c \\ (c,d)=1}} \int_{-\infty}^{\infty} c^{-k}\left(z+\frac{d}{c}\right)^{-k} e(-nz)\, e\left(\frac{ma}{c}\right) e\left(\frac{-m}{c^2\left(z+\frac{d}{c}\right)}\right) dz$$

$$z \Rightarrow z + d/c$$

$$= \delta_{mn} + \sum_{\substack{c>0 \\ d \bmod c \\ (c,d)=1}} c^{-k} \int_{-\infty}^{\infty} z^{-k} e\left(\frac{ma}{c}\right) e\left(-n\left(z-\frac{d}{c}\right)\right) e\left(\frac{-m}{c^2 z}\right) dz$$

$$= \delta_{mn} + \sum_{c>0} c^{-k} \underbrace{\sum_{\substack{d \bmod c \\ (c,d)=1}} e\left(\frac{ma+nd}{c}\right)}_{:= K(m,n;c)} \underbrace{\int_{-\infty}^{\infty} z^{-k} e\left(\frac{-m}{c^2 z} - nz\right) dz}_{I(m,n,c,k)}$$

$K(m,n;c) = \sum_{\substack{d \bmod c \\ (c,d)=1}} e\left(\frac{ma+nd}{c}\right)$ is called Kloosterman sum

Note $ad \equiv 1 \bmod c$, i.e. $a \equiv d^{-1} \bmod c$
and $K(m,n,c)$ can be written as

$\sum_{\substack{d \bmod c \\ (c,d)=1}} e\left(\frac{m\bar{d} + nd}{c}\right)$ where we denoted by $\bar{d}$ $d^{-1} \bmod c$.

$$q_m(n) = \delta_{mn} + \sum_{c>0} c^{-k} K(m,n,c)\, I(m,n,c,k)$$

The integral can be evaluated to obtain

$$I(m,n,c,k) = \begin{cases} 0 & \text{if } n \le 0 \\ \dfrac{(2\pi i)^k}{(k-1)!}(-n)^{k-1} & \text{if } m = 0 \\ (2\pi i)^k \left(c\sqrt{\dfrac{n}{m}}\right)^{k-1} J_{k-1}\left(\dfrac{4\pi\sqrt{mn}}{c}\right) & \text{if } m \end{cases}$$

Here $J_\nu(z)$ is the J-Bessel function

$$J_\nu(z) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!\,\Gamma(\ell+1+\nu)} \left(\frac{z}{2}\right)^{\nu+2\ell}.$$

If $m=0$, $\displaystyle a_0(n) = \frac{(2\pi i)^k}{(k-1)!} n^{k-1} \sum_{\substack{c>0 \\ d \bmod c \\ (c,d)=1}} c^{-k} e\left(\frac{nd}{c}\right)$

The sum $\displaystyle \sum_{\substack{d \bmod c \\ (c,d)=1}} e\left(\frac{nd}{c}\right) =: R_n(c)$ is called Ramanujan sum

One can show $\displaystyle \sum_{c=1}^{\infty} \frac{R_n(c)}{c^s} = \frac{\sigma_{s-1}(n)}{n^{s-1}\zeta(s)}$

(See Hardy-Wright, An intro. to number theory)

Hence $\displaystyle \sum_{c>0} c^{-k} R_n(c) = \frac{\sigma_{k-1}(n)}{n^{k-1}\zeta(k)}$

and

$\displaystyle a_0(n) = \frac{(2\pi i)^k}{(k-1)!} \frac{\sigma_{k-1}(n)}{\zeta(k)}$ is exactly the n-th coeff of E.

On the other hand

if $m \geq 1$ then $a_m(0) = 0$ since $\delta_{mn} = 0$, $I(m,0,c,k) = 0$

Hence Poincare series $P_m(z)$ for $m \geq 1$ are cusp forms

Prop 5.2 Let $P_m(z)$ be the mth Poincare series. Then $P_0^k(z) = E_k(z)$

and for $m \geq 0$

$$P_m(z) = \sum_{n=1}^{\infty} a_m(n) q^n \quad \text{with}$$

$$a_m(n) = \begin{cases} \delta_{mn} + \sum_{c>0} \dfrac{K(m,n,c)}{c} \dfrac{2\pi}{i^k} \left(\dfrac{n}{m}\right)^{\frac{k-1}{2}} J_{k-1}\left(\dfrac{4\pi\sqrt{mn}}{c}\right) & \text{if } n \geq 0,\ m \neq 0. \\ 1 & \text{if } m = n = 0. \\ \dfrac{(2\pi i)^k}{(k-1)!} \dfrac{\sigma_{k-1}(n)}{\zeta(k)} & \text{if } m = 0,\ n \neq 0. \end{cases}$$

Petersson Inner product: (5-9)

The Poincaré series also arises naturally via Linear Algebra. For this we first introduce an inner product on $S_k(\Gamma)$, which was defined by H. Petersson. To define this product we first introduce an invariant measure (inv. under $SL(2,\mathbb{R})$), on $\mathbb{H}$

Prop 5.3 Let $d\mu(z) = \dfrac{dx\,dy}{y^2}$, $(z = x+iy)$

Then $d\mu(\gamma z) = d\mu(z) \quad \forall \gamma \in SL(2,\mathbb{R})$

Pf. (1) Any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ can be written as a product of matrices of the form $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Indeed ($c \neq 0$) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & a/c \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1/c & 0 \\ 0 & c \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & d/c \\ 0 & 1 \end{pmatrix}$

Enough to check inv. under these special matrices using chain rule

or (2) If $w = \dfrac{az+b}{cz+d} = u+iv =: f(z)$

$z = x+iy$. Then f is holom. and the Jacobian of the change of variables from z to w is the matrix

$$J = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \underset{\text{Cauch-Riemann eqns}}{=} \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ -\frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} \end{pmatrix}$$

and $du\,dv = \|J\|\, dx\,dy$, $\|J\| = \det J = |f'(z)|^2$

If $w = \frac{az+b}{cz+d} = \gamma z$ then $\det J = \left|\frac{d}{dz}(\gamma z)\right|^2$

$= \left|1/(cz+d)^2\right|^2$

On the other hand

$$v = \operatorname{Im} w = \frac{y}{|cz+d|^2}$$

Hence $$\frac{du\,dw}{v^2} = \frac{\|J\|\, dx\,dy}{y^2/|cz+d|^4} = \frac{dx\,dy/|cz+d|^4}{y^2/|cz+d|^4}$$

$$= \frac{dx\,dy}{y^2}$$

We can now define an inner product on S_k

Defn Let $f, g \in S_k(\Gamma)$. The <u>Petersson Inner Product</u> of f and g is

$$\langle f, g \rangle := \int_{\Gamma \backslash H} f(z)\,\overline{g(z)}\, y^k\, d\mu(z)$$

Remark ① First note that if we let

$\phi(z) = f(z)\overline{g(z)}\, y^k$ then

$\phi(\gamma z) = \phi(z)$

since

$\phi(\gamma z) = f(\gamma z)\overline{g(\gamma z)}\,(\mathrm{Im}\,\gamma z)^k$

$= (cz+d)^k f(z)\,\overline{(cz+d)}^k\,\overline{g(z)}\,\dfrac{y^k}{|cz+d|^{2k}}$

$= \phi(z)$.

Hence $f(z)\overline{g(z)}\, y^k \dfrac{dx\,dy}{y^2}$ is invariant under Γ. Hence the integral does not depend on the choice of the fund. domain $\Gamma\backslash\mathbb{H}$ and is well defined.

② . Recall $f(z)y^{k/2}$ is bounded for $f \in S_k$. Hence $f(z)\overline{g(z)}\, y^k$ is bounded and the integral converges.

③ One can readily checks

ⓐ $\langle f, g\rangle = \overline{\langle g, f\rangle}$ ie $\langle\ ,\ \rangle$ is Hermitian

ⓑ $\langle f, f\rangle \geq 0$ and $= 0$ iff $f = 0$.

Hence $S_k(\Gamma)$ becomes an inner product space.

Recall every linear functional ψ on a finite-dim'l inner product space V is of the form $\varphi(v) = \langle v, v_\varphi \rangle$ for some

$v_\varphi \in V$. Now on the space $V = S_k(\Gamma)$, for every $n > 0$, we have a linear functional

$$\varphi_n : S_k(\Gamma) \longrightarrow \mathbb{C}$$
$$f \longrightarrow a_n$$

And the linear functions φ_n are essentially given by the Poincare series $P_n^k(z)$.

More precisely we have

<u>Thm 5.4</u> If $f \in S_k(\Gamma)$ and $P_m^k(z)$ is the mth Poincare series then

$$\langle f, P_m \rangle = \frac{\Gamma(k-1)}{(4\pi m)^{k-1}} a_m$$

where $f(z) = \sum a_m q^m$.

<u>Proof</u>: $\langle f, P_m \rangle = \int_{\Gamma \backslash H} f(z) \overline{P_m(z)} y^k \frac{dx\,dy}{y^2}$

$$= \int_{\Gamma \backslash H} f(z) \overline{\sum_{\gamma \in \Gamma_\infty \backslash \Gamma} e^{2\pi i m(\gamma z)} (cz+d)^{-k}} y^k \frac{dx\,dy}{y^2}$$

$$= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \int_{\Gamma \backslash H} f(z) e^{-2\pi i m(\overline{\gamma z})} \overline{j(\gamma, z)}^{-k} y^k \frac{dx\,dy}{y^2}$$

$z' = \gamma^{-1} z$.

$$= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \int_{\gamma(\Gamma \backslash H)} f(\gamma^{-1} z)\, e^{-2\pi i m \bar{z}}\, j(\gamma, \gamma^{-1} z)^{-k} (\operatorname{Im} \gamma^{-1} z)^k \frac{dx\,dy}{y^2}$$

Now $f(\gamma^{-1} z) = j(\gamma^{-1}, z)^k f(z)$

$$j(I, z) = j(\gamma\gamma^{-1}, z) = j(\gamma, \gamma^{-1} z)\, j(\gamma^{-1}, z)$$

Hence we get

$$\langle f, P_m \rangle = \underbrace{\sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \int_{\gamma(\Gamma \backslash H)}}\ \underbrace{f(z) e^{-2\pi i m \bar{z}}\, y^k \frac{dx\,dy}{y^2}}_{\text{indep of } \gamma}$$

"unfolding"

$$= \int_{\Gamma_\infty \backslash H} f(z)\, e^{-2\pi i m(x)} \cdot e^{-2\pi m y}\, y^k \frac{dx\,dy}{y^2}$$

Since for any set of reps $\{\gamma\} = R$ for $\Gamma_\infty \backslash \Gamma$ and for any fund domain $F = \Gamma \backslash H$ for Γ

$\bigcup_{\gamma \in R} \gamma F$ is a fund domain for Γ_∞.

$$\sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \int_{\gamma(\Gamma \backslash H)} = \sum_{\gamma \in R} \int_{\gamma F} = \int_{\bigcup_{\gamma \in R} (\gamma F)} = \int_{\Gamma_\infty \backslash H}$$

Since

$$H = \bigcup_{\gamma \in \Gamma} \gamma F = \bigcup_{\delta \in \Gamma_\infty} \bigcup_{\gamma \in \Gamma_\infty \backslash \Gamma} \delta\gamma F = \bigcup_{\delta \in \Gamma_\infty} \delta\left(\bigcup_{\gamma \in R} \gamma F\right)$$

Hence $\mathbb{H} = \Gamma_\infty \bigcup_{\gamma \in R} \gamma F$ and

$\bigcup_{\gamma \in R} \gamma F$ is a fund. domain for Γ_∞.

Ie $\Gamma_\infty \backslash \mathbb{H} = \bigcup_{\gamma \in R} (\gamma F)$

Hence $\langle f, P_m \rangle = \int_{\Gamma_\infty \backslash \mathbb{H}} f(z) \, e^{-2\pi i m x} e^{-2\pi m y} y^k \frac{dx\,dy}{y^2}$

Note once again the integrand is inv. under Γ_∞ hence indep of choice of fund. dom. for $\Gamma_\infty \backslash \mathbb{H}$. So we can choose

$\Gamma_\infty \backslash \mathbb{H}$ as the strip $\left\{ z \in \mathbb{H} \;\middle|\; \begin{array}{l} 0 < \operatorname{Re} z < 1 \\ \operatorname{Im} z > 0 \end{array} \right\}$

and obtain

$$\langle f, P_m \rangle = \int_0^\infty \int_0^1 \sum_n a_n e^{2\pi i n x} \cdot e^{-2\pi i m x} \cdot e^{-2\pi m y} y^k \frac{dx\,dy}{y^2}$$

$$= \sum_{n=1}^\infty a_n \int_0^\infty e^{-2\pi m y} y^k \frac{dy}{y^2} \underbrace{\int_0^1 e^{2\pi i (n-m) x} dx}_{\delta_{mn}}$$

$$= a_m \int_0^\infty e^{-4\pi m y} y^k \frac{dy}{y^2} = a_m \frac{\Gamma(k)}{(4\pi m)^{k-1}}$$

Cor 5.5 $k \geq 3$, $\{P_m^k(z) \mid m \geq 1\}$ generate $S_k(\Gamma)$, and

Pf let $\mathcal{U} = \text{span}\{P_m^k\} \subset S_k(\Gamma)$

and let $\mathcal{U}^\perp = \{f \in S_k(\Gamma) \mid \langle f, g \rangle = 0 \ \forall g \in \mathcal{U}\}$

$= \{f \in S_k(\Gamma) \mid \langle f, P_m \rangle = 0 \ \forall m \geq 1\}$

Take $f \in \mathcal{U}^\perp$ then

$$0 = \langle f, P_m^k \rangle = \frac{\Gamma(k-1)}{(4\pi m)^{k-1}} a_m = 0 \quad \forall m \geq 1$$

$\Rightarrow a_m = 0 \ \forall m \geq 1 \Rightarrow f \equiv 0$.

and $S_k(\Gamma) = \mathcal{U}$.

Remark: Note if $m = 0$, $P_0^k = E_k(z)$

and since $\langle f, P_0(z) \rangle \overset{?}{=} a_0 = 0$

$\langle f, E_k \rangle = 0$ i.e. E_k is orthogonal to $S_k(\Gamma)$. (Exercise)